
Multiscale BDDC for a saddle-point problem

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Abstract We propose a Multiscale BDDC for a class of saddle-point problems. The method solves for both flux and pressure variables. The fluxes are resolved in three-steps: the coarse solve is followed by subdomain solves, and last we look for a divergence-free flux correction and pressure variables using conjugate gradients with a Multilevel BDDC preconditioner. Because the coarse solve in the first step has the same structure as the original problem, we can use this procedure recursively and solve (a hierarchy of) coarse problems only approximately, utilizing the coarse problems known from the BDDC. The resulting algorithm thus first performs several upscaling steps, and then solves a hierarchy of problems that have the same structure but increase in size while sweeping down the levels, using the same components in the first and in the third step on each level, and also reusing the components from the higher levels. Because the coarsening can be quite aggressive, the number of outer iterations can be kept small and the additional computational cost is significantly reduced due to the reuse of the components. We also provide the condition number bound and numerical experiments confirming the theory.

Keywords Iterative substructuring · balancing domain decomposition · BDDC · multilevel methods · multiscale methods · saddle-point problems

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1 Introduction

The Balancing Domain Decomposition by Constraints (BDDC), proposed independently by Cros [4], Dohrmann [6], and Fragakis and Papadrakakis [11], is along with the Finite Element Tearing and Interconnecting - Dual, Primal (FETI-DP) method by Farhat et al. [8, 9] currently one of the most advanced and popular methods of iterative substructuring. These methods have been derived by modifications of the BDD method by Mandel [17], and of the FETI method by Farhat and Roux [10], respectively. The relations between these two families of methods have been studied extensively by many analysts in the substructuring field cf., e.g., [1, 16, 18, 20], and also [25]. The methods have been also extended to multiple levels: one can find multilevel extensions of the BDDC in [21, 26, 27, 32, 33] and of FETI in [14]. From our current point of view are important extensions to saddle-point problems, such as to the Stokes problem [15, 23, 24] and to the flow in porous media. One of the first domain decomposition methods for mixed finite element problems were proposed by Glowinski and Wheeler [12]. Their Method II has been preconditioned using BDD by Cowsar et al. [3] and using BDDC by Tu [31]. This approach is sometimes regarded as *hybrid* because the method iterates on a system of *dual* variables (as Lagrange multipliers) enforcing the continuity of flux variables across the substructure interfaces. However, in order to allow for a multilevel extension, we would like to retain the original *primal* variables, and therefore we find the recent work of Tu [29, 34] to be more relevant for our approach.

The multiscale finite element methods introduced by Hou and Wu [13] have evolved in the recent years into an active research area. The methods gained popularity due to the ability to model problems which contain many spatial scales such as composite materials or porous media. Their main idea is to construct multiscale finite element basis functions that are adaptive to the local property of the differential operator and provide a way to handle systems with extremely large numbers of degrees of freedom especially due to highly heterogeneous media. These methods also provide an important and interesting alternative to the empirical upscaling methods cf., e.g., an overview by Cushman et al. [5]. From this perspective our method can be viewed as a way of numerical upscaling via the coarse basis functions known from the BDDC. The construction of basis functions is fully decoupled and thus amenable to massively parallel computers. However, because the BDDC method and its multilevel version as such is now only a part of larger algorithmic framework, we have decided to call this new method as a Multiscale BDDC.

In this paper, we propose the Multiscale BDDC for a class of saddle-point problems. Our starting point is the algorithm of Ewing and Wang [7], see also Mathew [22]. The basic idea is to solve for flux variables in three-steps: first we perform a coarse solve which is followed by independent subdomain solves with zero boundary conditions in the second step. In the third step, we look for a flux correction and pressures. Due to the design of the algorithm, the flux correction is divergence-free, and we can use conjugate gradients (CG, resp. PCG) with a Multilevel BDDC preconditioner that preserves all of the iterates

in the divergence-free subspace. Applications of two-, resp. three-level BDDC in the third step of this algorithm have been studied by Tu in [29, 34]. Also, one has to make a careful decision in the design of the coarse solve for the first step. A straightforward idea is to use the same, but “coarse” finite element discretization and a natural (linear) interpolation between the two meshes as considered in [22, 29]. Alternatively, the coarse solve has been obtained by an action of the BDDC preconditioner on a carefully chosen vector by Tu in [30, 34] and she has also numerically observed a very similar performance of the two choices [30, Section 4.8]. Obviously, we will favor the second idea. Next, noting that the coarse solve in the first step has the same structure as the original problem, we can use the algorithm recursively, and solve a hierarchy of coarse solves only approximately. The resulting algorithm thus first creates a hierarchy of problems with similar structure scaling-up through the levels. Then this hierarchy is solved, while sweeping down the levels in a loop of outer iterations, using the same components in the first and the third step on each level, and also reusing the components from all of the previous (higher) levels. Because the coarsening can be quite aggressive, the number of outer iterations can be kept small and the additional computational cost is significantly reduced due to the reusing of components. Unlike some of the previous works, we do not use the global partially assembled matrices neither the change of variables which seem to require more work in the present settings.

It is important to note that for the solution of closely related Stokes problem, the algorithm is reduced to step three because the solution itself is divergence-free. We also remark that the present approach is limited by a special choice of finite elements. In particular, we will work with the lowest-order Raviart-Thomas (RT0) elements that have piecewise constant basis functions for pressure variables. This is not the case when, e.g., Taylor-Hood elements are used and the BDDC preconditioned operator is no longer invariant on the divergence-free subspace [24]. Finally, we note that our framework allows for irregular mesh decompositions, heterogeneous coefficients possibly utilizing the adaptive approach as in [19, 26], and also allows for a relatively straightforward extension into 3D. However, such extensions will be studied elsewhere.

The paper is organized as follows. In Section 2 we introduce the model problem, in Section 3 we introduce its mixed finite element discretization and recall the original algorithm of Ewing and Wang. In Section 4 we derive the two-scale version of this algorithm using the BDDC components. In Section 5 we formulate the Multiscale BDDC method. In Section 6 we derive the condition number bound for the model problem, and finally in Section 7 we report on numerical experiments with a particular application to flow in porous media.

Throughout the paper we find it more convenient to work with abstract finite dimensional spaces and linear operators between them instead of the space \mathbb{R}^n and matrices. The results can be easily converted to the matrix language by choosing a finite element basis. For a symmetric positive definite bilinear form a , we will denote the energy norm by $\|u\|_a = \sqrt{a(u, u)}$.

2 Model problem

Let Ω be a bounded polygonal domain in \mathbb{R}^n , $n = 2$. We are interested in a solution of the following scalar, second-order, elliptic problem

$$-\nabla \cdot k \nabla p = f \quad \text{in } \Omega, \quad (1)$$

where k is a symmetric, uniformly positive definite matrix with bounded coefficients, the right-hand side $f \in L^2(\Omega)$, subject to sufficiently smooth boundary data on $\partial\Omega = \Gamma_D \cup \Gamma_N$. The variable p will be called pressure.

Introducing the so-called flux variable

$$\mathbf{u} = -k \nabla p, \quad (2)$$

we may rewrite (1) as a first-order system, generally known as Darcy's problem,

$$\begin{aligned} k^{-1} \mathbf{u} + \nabla p &= 0 \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= f \quad \text{in } \Omega, \\ p &= g_N \quad \text{on } \Gamma_N, \\ \mathbf{u} \cdot \mathbf{n} &= g_D \quad \text{on } \Gamma_D, \end{aligned}$$

where \mathbf{n} is the unit outward normal of Ω , and for the the boundary conditions it holds that $g_N \in H^{1/2}(\Gamma_N)$, and $g_D \in H_{00}^{-1/2}(\Gamma_D)$.

Let us define a space

$$\mathbf{H}(\Omega; \text{div}) = \left\{ \mathbf{v} \in (L^2(\Omega))^n; \nabla \cdot \mathbf{v} \in L^2(\Omega); \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \cap \Gamma_D \right\},$$

and denoting by H_Ω the characteristic size of Ω , the norm is defined as

$$\|v\|_{\mathbf{H}(\Omega; \text{div})}^2 = \|\mathbf{v}\|_{L^2(\Omega)}^2 + H_\Omega^2 \|\nabla \cdot \mathbf{v}\|_{L^2(\Omega)}^2.$$

The weak form of the Darcy's problem is

$$\int_{\Omega} k^{-1} \mathbf{u} \cdot \mathbf{v} \, dx - \int_{\Omega} p \nabla \cdot \mathbf{v} \, dx = \int_{\Gamma_N} g_N \mathbf{v} \cdot \mathbf{n} \, ds, \quad \forall \mathbf{v} \in \mathbf{H}(\Omega; \text{div}), \quad (3)$$

$$- \int_{\Omega} \nabla \cdot \mathbf{u} q \, dx = - \int_{\Omega} f q \, dx, \quad \forall q \in L^2(\Omega). \quad (4)$$

We will consider the case $\Gamma_N = \emptyset$, so that the right-hand side term in (3) will vanish, which requires the compatibility condition

$$\int_{\Omega} f \, dx + \int_{\partial\Omega} g_D \, ds = 0, \quad (5)$$

and the pressure p will be determined uniquely up to an additive constant. We refer to the monographs [2, 28] for additional details.

3 Mixed finite elements and basic algorithm

Let U be the lowest order Raviart-Thomas finite element space with a zero normal component on $\partial\Omega$ and Q be a space of piecewise constants with a zero mean on Ω . These two spaces, defined on the triangulation \mathcal{T}_h of Ω where h denotes the mesh size, are finite dimensional subspaces of $\mathbf{H}(\Omega; \text{div})$ and $L^2(\Omega)$, respectively, and they satisfy a uniform inf-sup condition, see [2].

In the mixed variational formulation of the Darcy's problem, cf. eq. (3)-(4), we would like to find a pair $(u, p) \in (U, Q)$ such that

$$a(u, v) + b(v, p) = 0, \quad \forall v \in U, \quad (6)$$

$$b(u, q) = \langle f, q \rangle, \quad \forall q \in Q. \quad (7)$$

Let us split the domain Ω into non-overlapping subdomains Ω_i , $i = 1, \dots, N$. Accordingly, let us split the solution spaces as

$$U = U_0 + (\oplus_{i=1}^N U_i) + U_{\text{corr}}, \quad (8)$$

$$Q = \oplus_{i=0}^N Q_i. \quad (9)$$

The spaces U_0 , Q_0 are obtained by considering subdomains for a moment as macroelements, and the spaces U_i , Q_i , for $i = 1, \dots, N$, are obtained by a restriction from the global solution spaces U , Q . The introduction of the auxiliary space $U_{\text{corr}} \subset U$ is motivated by an observation that in general

$$U \neq U_0 + (\oplus_{i=1}^N U_i), \quad (10)$$

because the fluxes on subdomain interfaces might not be constant. Also, in order to write $U_I = \oplus_{i=1}^N U_i$ we require zero fluxes over the subdomain interfaces. Finally, to determine the pressure p uniquely, we will consider the component $p_0 \in Q_0$, which is constant in each subdomain Ω_i , to have a zero average over the whole domain Ω , and the components $p_i \in Q_i$ to have zero averages over the subdomain Ω_i and identically equal to zero in other subdomains. We note that we will take an advantage of this splitting, in particular because for all $u_I \in U_I$ and $q_0 \in Q_0$, it holds, by the divergence theorem, that

$$b(u_I, q_0) = - \int_{\Omega} (\nabla \cdot u_I) q_0 = 0. \quad (11)$$

The following algorithm is due to Ewing and Wang [7], cf. also Mathew [22].

Algorithm 1 Find the solution $(u, p) \in (U, Q)$ of the problem (6)-(7) as

$$u = u_0 + \sum_{i=1}^N u_i + u_{\text{corr}},$$

in the following three steps: Compute

1. the coarse component $(u_0, p_0) \in (U_0, Q_0)$ by solving

$$a(u_0, v_0) + b(v_0, p_0) = 0, \quad \forall v_0 \in U_0, \quad (12)$$

$$b(u_0, q_0) = \langle f, q_0 \rangle, \quad \forall q_0 \in Q_0. \quad (13)$$

Note that because $Q_0 \subsetneq Q$, in general

$$b(u_0, q) \neq \langle f, q \rangle, \quad \forall q \in Q.$$

2. the substructure components $(u_i, p_i) \in (U_i, Q_i)$ for $i = 1, \dots, N$ from

$$\begin{aligned} a(u_i, v_i) + b(v_i, p_i) &= -a(u_0, v_i), \quad \forall v_i \in U_i, \\ b(u_i, q_i) &= \langle f, q_i \rangle - b(u_0, q_i), \quad \forall q_i \in Q_i. \end{aligned}$$

Add the computed solutions as

$$u^* = u_0 + \sum_{i=1}^N u_i.$$

Due to the correction in the second step, and with respect to (9), we obtain

$$b(u^*, q) = \langle f, q \rangle \quad \forall q \in Q. \quad (14)$$

On the other hand, from (10), in general $u^* \neq u$. Therefore, we also need

3. the correction $u_{\text{corr}} \in U_{\text{corr}} \subset U$. Considering

$$u = u^* + u_{\text{corr}},$$

substituting into (6)-(7) and using (14), compute (u_{corr}, p) from

$$\begin{aligned} a(u_{\text{corr}}, v) + b(v, p) &= -a(u^*, v), \quad \forall v \in U, \\ b(u_{\text{corr}}, q) &= 0, \quad \forall q \in Q. \end{aligned}$$

Remark 1 Because the pressure components p_0, p_I computed in the first two steps are tested only against proper subspaces of U , we simply disregard them.

The application of the BDDC preconditioner for the computation of u_{corr} for the two-, resp. three-level BDDC method has been studied by Tu [29, 34]. However, comparing (12)-(13) with (6)-(7), we see that in fact we can use the same algorithm recursively, with multiple scales, to solve for both u_0 and u_{corr} . But first, let us reformulate Algorithm 1 as the two-scale BDDC method.

4 Two-scale BDDC

We begin by introducing the substructuring components. Let Ω be decomposed into nonoverlapping subdomains Ω_i , $i = 1, \dots, N$, also called substructures, forming a quasi-uniform triangulation of Ω with the characteristic subdomain size H . Each substructure is a union of the lowest order Raviart-Thomas (RT0) finite elements with a matching discretization across the substructure interfaces. Let $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$ be the interface of the substructure Ω_i and let $\Gamma = \cup_{i=1}^N \Gamma_i$. Let us denote by \mathcal{F} the set of all faces between substructures, i.e., in the present context the set of all intersections $\Gamma_{ij} = \Gamma_i \cap \Gamma_j$, $i \neq j$ such that $\text{meas}(\Gamma_{ij}) > 0$. Note that with respect to our discretization we define only *faces*, but no *corners* (nor *edges* in 3D) known from other types of substructuring. Let us also slightly generalize setting of coefficient k by allowing for coefficients k_i which are symmetric and uniformly elliptic in each subdomain separately.

The first step in substructuring is typically reduction of the problem to interfaces. Because only the flux variables are associated with the interfaces, we will introduce additional spaces and operators related only to the space U . For this reason we can for the moment closely follow our previous work [21].

Let W_i be the space of flux finite element functions on a substructure Ω_i such that all of their degrees of freedom on $\partial\Omega_i \cap \partial\Omega$ are zero, and let

$$W = W_1 \times \dots \times W_N.$$

Now $U \subset W$ can be viewed as the subspace of all function from W continuous across substructure interfaces. Define $U_I \subset U$ as the subspace of functions that are zero on the interface Γ , i.e., the space of “interior” functions. Denote by P the energy orthogonal projection from W onto U_I ,

$$P : w \in W \mapsto v_I \in U_I : a(v_I, z_I) = a(w, z_I), \quad \forall z_I \in U_I.$$

Functions from $(I - P)W$, i.e., from the nullspace of P , are called discrete harmonic; these functions are a -orthogonal to U_I and they are energy minimal with respect to increments in U_I . Next, let \widehat{W} be the space of all discrete harmonic functions that are continuous across substructure interfaces, that is

$$\widehat{W} = (I - P)U, \tag{15}$$

and in particular,

$$U = U_I \oplus \widehat{W}, \quad U_I \perp_a \widehat{W}. \tag{16}$$

The BDDC method is a two-level preconditioner characterized by the selection of certain *coarse degrees of freedom*. In the present setting, the value of a coarse degree of freedom on a face will be taken as the average of the fine scale degrees of freedom on the same face. Let $\widetilde{W} \subset W$ be the subspace of all functions such that the values of any coarse degrees of freedom have a common value and vanish on $\partial\Omega$. We assume that

$$a \text{ is positive definite on } \widetilde{W}. \tag{17}$$

Define $\widetilde{W}_\Pi \subset \widetilde{W}$ as the subspace of all functions such that their coarse degrees of freedom between adjacent substructures coincide, and such that their energy is minimal, and let us also define $\widetilde{W}_\Delta \subset \widetilde{W}$ as the subspace of all function such that their coarse degrees of freedom vanish. Clearly, functions in \widetilde{W}_Π are uniquely determined by the values of their coarse degrees of freedom, and

$$\widetilde{W} = \widetilde{W}_\Delta \oplus \widetilde{W}_\Pi, \quad \text{and} \quad \widetilde{W}_\Delta \perp_a \widetilde{W}_\Pi. \quad (18)$$

Let E be a projection from \widetilde{W} onto U , defined by taking some weighted average of corresponding degrees of freedom on substructure interfaces. Finally, let us define as before, cf. eq. (9),

$$Q = Q_0 \oplus Q_I,$$

where Q_0 consists of constant functions in each subdomain, such that

$$\int_{\Omega} Q_0 = 0 \quad \text{and} \quad \int_{\Omega_i} Q_I = 0, \quad i = 1, \dots, N.$$

We remark that the restriction to Q_I can be enforced by “bubble” constraints, and the basis for Q_0 can be obtained from the one of Q with only a little extra work. In the following, we will also need a subspace of *balanced* functions

$$\widetilde{W}_B = \left\{ v \in \widetilde{W} : b(v, q_0) = 0 \quad \forall q_0 \in Q_0 \right\},$$

for which we get the equivalence, cf. [28, Section 9.4], and also [22, eq. (15)],

$$c \|v\|_a^2 \leq \|v\|_{\mathbf{H}(\Omega; \text{div})}^2 \leq C \|v\|_a^2, \quad \forall (v, q_0) \in (\widetilde{W}_B, Q_0). \quad (19)$$

Next, observe that it is only required for u^* to satisfy (14). In particular, we do not need the substructures to form the same discretization as on the finite element level. Instead, we can conveniently retain the algebraic framework of the BDDC method introduced above and use its coarse problem in place of the coarse solve in Step 1. Specifically, let us set $U_0 = \widetilde{W}_\Pi$. We are now ready to take the second look at Algorithm 1 and formulate its first modification.

Algorithm 2 (Two-scale BDDC) Find the solution $(u, p) \in (U, Q)$ of the problem (6)-(7) by computing:

1. the coarse component $u_0 \in \widehat{W}$ solving for $\tilde{w}_0 \in \widetilde{W}_\Pi$ the system

$$a(\tilde{w}_0, \tilde{v}_\Pi) + b(\tilde{v}_\Pi, p_0) = 0, \quad \forall \tilde{v}_\Pi \in \widetilde{W}_\Pi, \quad (20)$$

$$b(\tilde{w}_0, q_0) = \langle f, q_0 \rangle, \quad \forall q_0 \in Q_0, \quad (21)$$

and applying the projection

$$u_0 = E\tilde{w}_0.$$

2. the substructure components $u_I \in U_I$ solving

$$\begin{aligned} a(u_I, v_I) + b(v_I, p_I) &= -a(u_0, v_I), \quad \forall v_I \in U_I, \\ b(u_I, q_I) &= \langle f, q_I \rangle - b(u_0, q_I), \quad \forall q_I \in Q_I, \end{aligned}$$

and combining the solutions $u^* = u_0 + u_I$.

3. the correction and the pressure $(u_{\text{corr}}, p) \in (U, Q)$ from

$$\begin{aligned} a(u_{\text{corr}}, v) + b(v, p) &= -a(u^*, v), \quad \forall v \in U, \\ b(u_{\text{corr}}, q) &= 0, \quad \forall q \in Q. \end{aligned}$$

Specifically, use the PCG method with the two-level BDDC preconditioner defined in Algorithm 3.

Finally, combine the three solutions as

$$u = u_0 + u_I + u_{\text{corr}}.$$

Note that we again disregard the pressures p_0 and p_I from Steps 1 and 2 as in Algorithm 1. The algorithm of the two-level BDDC preconditioner used in Step 3 is closely related to the original version for elliptic problems, cf. [21, Algorithm 11]. For completeness its version for saddle-point problems follows.

Algorithm 3 (Two-level BDDC preconditioner) Define the preconditioner $(r, 0) \in (U', Q') \mapsto (u, p) \in (U, Q)$ as follows:

Compute the interior pre-correction $(u_I, p_I) \in (U_I, Q_I)$ from

$$\begin{aligned} a(u_I, z_I) + b(z_I, p_I) &= \langle r, z_I \rangle, \quad \forall z_I \in U_I, \\ b(u_I, q_I) &= 0, \quad \forall q_I \in Q_I. \end{aligned}$$

Set up the updated residual

$$r_B \in U', \quad \langle r_B, v \rangle = \langle r, v \rangle - [a(u_I, v) + b(v, p_I)], \quad \forall v \in U.$$

Compute the substructure correction $w_\Delta \in \widetilde{W}_\Delta$ from

$$\begin{aligned} a(w_\Delta, z_\Delta) + b(z_\Delta, p_{I\Delta}) &= \langle r_B, Ez_\Delta \rangle, \quad \forall z_\Delta \in \widetilde{W}_\Delta, \\ b(w_\Delta, q_I) &= 0, \quad \forall q_I \in Q_I. \end{aligned}$$

Compute the coarse correction $(w_\Pi, p_0) \in (\widetilde{W}_\Pi, Q_0)$ from

$$\begin{aligned} a(w_\Pi, z_\Pi) + b(z_\Pi, p_0) &= \langle r_B, Ez_\Pi \rangle, \quad \forall z_\Pi \in \widetilde{W}_\Pi, \\ b(w_\Pi, q_0) &= 0, \quad \forall q_0 \in Q_0. \end{aligned}$$

Add the averaged corrections

$$u_B = E(w_\Delta + w_\Pi).$$

Compute the interior post-correction $(v_I, q_I) \in (U_I, Q_I)$ from

$$\begin{aligned} a(v_I, z_I) + b(z_I, q_I) &= a(u_B, z_I), \quad \forall z_I \in U_I, \\ b(v_I, \bar{q}_I) &= b(u_B, \bar{q}_I), \quad \forall \bar{q}_I \in Q_I. \end{aligned}$$

Apply the combined corrections

$$\begin{aligned} u &= u_I + u_B - v_I, \\ p &= p_I + p_0 - q_I. \end{aligned}$$

Remark 2 The solve in the space \widetilde{W}_Δ gives rise to independent problems on substructures and the global coarse problem in the space \widetilde{W}_Π is exactly the same as the one used in Step 1 of Algorithm 2.

Clearly, one could implement Step 3 of Algorithm 2 by eliminating first the interior unknowns in (U_I, Q_I) , i.e., performing the static condensation, iteratively solving the problem in the spaces (\widetilde{W}, Q_0) , and recovering the interiors after the convergence. This would remove the interior pre-, and post-corrections from Algorithm 3, and the output of the preconditioner would not be required to be divergence-free but only balanced, cf. [28, Section 9.4]; such approach might be also more appealing from the practical point of view. For a proof that given a sufficient number of constraints, the PCG method with the two-level BDDC preconditioner is invariant on the space of balanced, resp. divergence-free functions see [29, Lemma 2] or Lemma 1 in the next section.

Due to the splitting of the spaces, in particular by relaxing only the velocity components in the action of the preconditioner, and with respect to the equivalence of norms (19), we can conveniently use the a -norm in the following estimate, cf., e.g., [28, Lemma 9.10] for more detailed discussion.

Theorem 4 ([29, Theorem 1]) *The condition number κ of the two-level BDDC preconditioner from Algorithm 3 satisfies the bound*

$$\kappa \leq \omega = \max \left\{ \sup_{w \in \widetilde{W}_B} \frac{\|(I - P)Ew\|_a^2}{\|w\|_a^2}, 1 \right\} \leq C \left(1 + \log \frac{H}{h} \right)^2. \quad (22)$$

Remark 3 In [29, Lemma 8], the supremum was taken over the space $(I - P)\widetilde{W}_B$ of discrete harmonic balanced function. Nevertheless, the bound remains the same by considering the larger space \widetilde{W}_B , cf. [21, Remark 16] for details.

In Algorithm 2, the coarse problem used in Steps 1 and 3 is solved exactly, and therefore becomes a bottleneck in the case of many substructures. In the next section we will suggest its further modification by using it recursively for Step 1 on a multiple of different scales, leading to the Multiscale BDDC.

5 Multiscale BDDC

We extend Algorithm 2 to multiple scales by using it recursively for Step 1, leading to a multilevel decomposition, and introducing thus a loop of outer iterations with the size given by the number of different decomposition scales.

The substructuring components from Section 4 will be denoted by an additional superscript ¹, as Ω_i^1 , $i = 1, \dots, N^1$, etc., and called level 1. The level 1 coarse problem solved in (20)-(21) will be called the level 2 problem. It has the same finite element structure as the original problem (6)-(7) on level 1, so we put $\widehat{W}_H^1 = U^2$ and $Q_0^1 = Q^2$. Level 1 substructures are level 2 elements and level 1 coarse degrees of freedom are level 2 degrees of freedom. Repeating this process recursively, level $\ell - 1$ substructures become level ℓ elements, and the level ℓ substructures are agglomerates of level ℓ elements. Level ℓ substructures are denoted by Ω_i^ℓ , $i = 1, \dots, N^\ell$, and they are assumed to form a conforming triangulation with a characteristic substructure size H^ℓ . For convenience, we denote by Ω_i^0 the original finite elements and put $H^0 = h$. The interface Γ^ℓ on level ℓ is defined as the union of all level ℓ boundary nodes, i.e., nodes shared by at least two level ℓ substructures, and we note that $\Gamma^\ell \subset \Gamma^{\ell-1}$. Level $\ell - 1$ coarse degrees of freedom become level ℓ degrees of freedom. The shape functions on level ℓ are determined by minimization of energy with respect to level $\ell - 1$ shape functions, subject to the value of exactly one level ℓ degree of freedom being one and others level ℓ degrees of freedom being zero. The minimization is done on each level ℓ element (level $\ell - 1$ substructure) separately, so the values of level $\ell - 1$ degrees of freedom are in general discontinuous between level $\ell - 1$ substructures, and only the values of level ℓ degrees of freedom between neighboring level ℓ elements coincide.

The development of the spaces on level ℓ now parallels the finite element setting in Section 4. Denote $U^\ell = \widehat{W}_H^{\ell-1}$. Let W_i^ℓ be the space of functions on the substructure Ω_i^ℓ , such that all of their degrees of freedom on $\partial\Omega_i^\ell \cap \partial\Omega$ are zero, and let

$$W^\ell = W_1^\ell \times \dots \times W_{N^\ell}^\ell.$$

Now $U^\ell \subset W^\ell$ can be viewed as the subspace of all functions from W^ℓ that are continuous across the interface Γ^ℓ . Define $U_I^\ell \subset U^\ell$ as the subspace of functions that are zero on Γ^ℓ , i.e., the functions “interior” to the level ℓ substructures. Denote by P^ℓ the energy orthogonal projection from W^ℓ onto U_I^ℓ . Denote by \widehat{P}^ℓ the energy orthogonal projection from W^ℓ onto U^ℓ . Functions from $(I - P^\ell)W^\ell$, i.e., from the nullspace of P^ℓ , are called discrete harmonic on level ℓ ; these functions are a -orthogonal to U_I^ℓ and energy minimal with respect to increments in U_I^ℓ . Next, let \widehat{W}^ℓ be the space of discrete harmonic functions that are continuous across substructure interfaces on level ℓ , that is

$$\widehat{W}^\ell = (I - P^\ell)U^\ell,$$

and in particular,

$$U^\ell = U_I^\ell \oplus \widehat{W}^\ell, \quad U_I^\ell \perp_a \widehat{W}^\ell. \quad (23)$$

Let $\widetilde{W}^\ell \subset W^\ell$ be the subspace of all functions such that the values of any coarse degrees of freedom on level ℓ have a common value for all relevant level ℓ substructures and vanish on $\partial\Omega_i^\ell \cap \partial\Omega$. Define $\widetilde{W}_H^\ell \subset \widetilde{W}^\ell$ as the subspace of all functions such that their level ℓ coarse degrees of freedom between adjacent substructures coincide, and such that their energy is minimal, i.e.,

$$\widetilde{W}_H^\ell = (I - P^\ell) \widetilde{W}_H^\ell, \quad (24)$$

and for completeness let us also define $\widetilde{W}_\Delta^\ell \subset W^\ell$ as the subspace of all functions such that their level ℓ coarse degrees of freedom vanish. Clearly, functions in \widetilde{W}_H^ℓ are uniquely determined by the values of their level ℓ coarse degrees of freedom, and

$$\widetilde{W}_\Delta^\ell \perp_a \widetilde{W}_H^\ell, \quad \widetilde{W}^\ell = \widetilde{W}_\Delta^\ell \oplus \widetilde{W}_H^\ell. \quad (25)$$

Let E^ℓ be a projection from \widetilde{W}^ℓ onto U^ℓ , defined by taking some weighted average on Γ^ℓ . Finally, let us define as before, cf. eq. (9),

$$Q^\ell = Q_0^\ell \oplus Q_I^\ell, \quad (26)$$

where Q_0^ℓ consists of constant functions in each level ℓ substructure, such that

$$\int_{\Omega^\ell} Q_0^\ell = 0 \quad \text{and} \quad \int_{\Omega_i^\ell} Q_I^\ell = 0, \quad i = 1, \dots, N^\ell.$$

Let us also first define, for levels $\ell = 1, \dots, L-1$, a hierarchy of *balanced* spaces

$$\begin{aligned} \widetilde{W}_B^\ell &= \left\{ w \in \widetilde{W}^\ell : b(w, q_0) = 0 \quad \forall q_0 \in Q_0^\ell \right\}, \\ \widehat{W}_B^\ell &= \left\{ w \in \widehat{W}^\ell : b(w, q_0) = 0 \quad \forall q_0 \in Q_0^\ell \right\}. \end{aligned}$$

We are now ready to generalize the two-scale Algorithm 2 to multiple scales.

Algorithm 5 (Multiscale BDDC) Find the solution $(u^1, p^1) \in (U^1, Q^1)$ of the problem (6)-(7) in the following steps:

for $\ell = 1, \dots, L-1$,

If $\ell = L-1$, set-up and solve (by a direct solve) the coarse problem for u_0^{L-1} .
Else, set-up level $\ell+1$ decomposition and formulate the scale $\ell+1$ problem.

end

for $\ell = L-1, \dots, 1$,

Find the substructure components $u_I^\ell \in U_I^\ell$
and combine the two solutions $u^{*,\ell} = u_0^\ell + u_I^\ell$.
Find the correction $u_{\text{corr}}^\ell \in U^\ell$ and $p^\ell \in Q^\ell$ (we keep p^ℓ only if $\ell = 1$), using the PCG with the Multilevel BDDC preconditioner from Algorithm 6.
Combine the three solutions as $u^\ell = u_0^\ell + u_I^\ell + u_{\text{corr}}^\ell$.
If $\ell > 1$, set $u_0^{\ell-1} = u^\ell$ (the coarse component for the next iteration).

end

We note that the first loop provides a natural approach of scaling-up through the levels. The Multilevel BDDC preconditioner used in Step 3 of Algorithm 5 consists of recursive application of the two-level BDDC preconditioner for the approximate solution of the hierarchy of the coarse problems that were pre-computed in Step 1. Even though the preconditioner differs only little from its original version for elliptic problems described in [21, Algorithm 17], we again include its saddle-point version here for completeness.

Algorithm 6 (Multilevel BDDC preconditioner) *Define the preconditioner $(r^\ell, 0) \in (U^{\ell'}, Q^{\ell'}) \mapsto (u^\ell, p^\ell) \in (U^\ell, Q^\ell)$ as follows:*
for $k = \ell, \dots, L - 1$,

Compute the interior pre-correction $(u_I^k, p_I^k) \in (U_I^k, Q_I^k)$ from

$$a(u_I^k, z_I^k) + b(z_I^k, p_I^k) = \langle r^k, z_I^k \rangle, \quad \forall z_I^k \in U_I^k, \quad (27)$$

$$b(u_I^k, q_I^k) = 0, \quad \forall q_I^k \in Q_I^k. \quad (28)$$

Set up the updated residual

$$r_B^k \in U^{k'}, \quad \langle r_B^k, v^k \rangle = \langle r_B^k, v^k \rangle - [a(u_I^k, v^k) + b(v^k, p_I^k)], \quad \forall v^k \in U^k.$$

Compute the substructure correction $w_\Delta^k \in \widetilde{W}_\Delta^k$ from

$$a(w_\Delta^k, z_\Delta^k) + b(z_\Delta^k, p_{I\Delta}^k) = \langle r_B^k, E^k z_\Delta^k \rangle, \quad \forall z_\Delta^k \in \widetilde{W}_\Delta^k, \\ b(u_I^k, q_I^k) = 0, \quad \forall q_I^k \in Q_I^k.$$

Compute the coarse correction $(w_\Pi^k, p_0^k) \in (\widetilde{W}_\Pi^k, Q_0^k)$ from

$$a(w_\Pi^k, z_\Pi^k) + b(z_\Pi^k, p_0^k) = \langle r_B^k, E^k z_\Pi^k \rangle, \quad \forall z_\Pi^k \in \widetilde{W}_\Pi^k, \quad (29)$$

$$b(w_\Pi^k, q_0^k) = 0, \quad \forall q_0^k \in Q_0^k. \quad (30)$$

If $k = L - 1$, solve the coarse problem directly and set

$$u^L = w_\Pi^{L-1}, \\ p^L = p_0^{L-1},$$

else, set $U^{k+1} = \widetilde{W}_\Pi^{k'}$ and set up the right-hand side for level $k + 1$,

$$r^{k+1} \in U^{k+1'}, \quad \langle r^{k+1}, z^{k+1} \rangle = \langle r_B^k, E^k z^{k+1} \rangle, \quad \forall z^{k+1} \in U^{k+1},$$

end

for $k = L - 1, \dots, \ell$,

Average the approximate corrections,

$$u_B^k = E^k (w_\Delta^k + u^{k+1}), \quad (31)$$

$$p_0^k = p^{k+1}. \quad (32)$$

Compute the interior post-correction $(v_I^k, q_I^k) \in (U_I^k, Q_I^k)$ from

$$a(v_I^k, z_I^k) + b(z_I^k, q_I^k) = a(u_B^k, z_I^k), \quad \forall z_I^k \in U_I^k, \quad (33)$$

$$b(v_I^k, \bar{q}_I^k) = b(u_B^k, \bar{q}_I^k), \quad \forall \bar{q}_I^k \in Q_I^k. \quad (34)$$

Apply the combined corrections,

$$\begin{aligned} u^k &= u_I^k + u_B^k - v_I^k, \\ p^k &= p_I^k + p_0^k - q_I^k. \end{aligned}$$

end

In order to guarantee that the Multilevel BDDC preconditioner is invariant on the space of divergence-free functions, we will need the following:

Assumption 7 *Let there exist a sufficient number of constraints as coarse degrees of freedom, such that on any decomposition level ℓ it holds that*

$$\begin{aligned} E^\ell w_\Delta &= 0, \quad \forall w_\Delta \in \widetilde{W}_\Delta^\ell, \\ E^\ell w_\Pi &= w_\Pi, \quad \forall w_\Pi \in \widetilde{W}_\Pi^\ell. \end{aligned}$$

Remark 4 In implementation, we can quite easily satisfy Assumption 7 by prescribing coarse degrees of freedom as averages over every face on every decomposition level. Then the values of coarse degrees of freedom of functions from the space $\widetilde{W}_\Delta^\ell$ are zero, and the values of coarse degrees of freedom for functions from the space \widetilde{W}_Π^ℓ for all (pairs of) adjacent substructures coincide.

Lemma 1 *Let Assumption 7 be satisfied. Then, for any $w \in \widetilde{W}_B^\ell$, it also holds that $(I - P^\ell) E^\ell w \in \widetilde{W}_B^\ell$ and the output of the Multilevel BDDC preconditioner in Algorithm 6 is divergence-free.*

Proof First, let us show that given $w \in \widetilde{W}^\ell$, for all $q_0 \in Q_0^\ell$, $\ell = 1, \dots, L-1$,

$$b(w, q_0) = 0 \quad \Rightarrow \quad b((I - P^\ell) E^\ell w, q_0) = 0.$$

Using (25), $w = w_\Delta + w_\Pi$. Then, by Assumption 7 and (24), we obtain

$$b((I - P^\ell) E^\ell w_\Delta, q_0) + b((I - P^\ell) E^\ell w_\Pi, q_0) = b(w_\Pi, q_0) = 0,$$

which shows that $(I - P^\ell) E^\ell w \in \widetilde{W}_B^\ell$. Next, note (23) and (26). In particular, on any level ℓ and for any $u_I \in U_I^\ell$ we have by (11) that

$$b(w_\Pi + u_I, q_0) = b(w_\Pi, q_0) = 0, \quad \forall q_0 \in Q_0^\ell,$$

and using (34), the interior post-correction satisfies

$$b(w_\Pi + u_I, q_I) = 0, \quad \forall q_I \in Q_I^\ell.$$

Finally, using (27) and (29) implies that u^k is divergence-free. \square

Thus with a careful choice of the initial guess, the output of the Multilevel BDDC preconditioner is divergence-free and by induction all the PCG iterates, which are linear combinations of the initial error and the outputs of the preconditioner, stay in the divergence-free subspace.

Finally, using the equivalence (19), Lemma 1, and also with respect to Remark 3, the following variant of [21, Lemma 20] carries over.

Lemma 2 *If for some $\omega^\ell \geq 1$,*

$$\|(I - P^\ell)E^\ell w^\ell\|_a^2 \leq \omega^\ell \|w^\ell\|_a^2, \quad \forall w^\ell \in \widetilde{W}_B^\ell, \quad \ell = 1, \dots, L-1, \quad (35)$$

then the Multilevel BDDC preconditioner satisfies $\kappa \leq \prod_{\ell=1}^{L-1} \omega^\ell$.

6 Condition number bound for the model problem

We will now derive a condition number bound for the model problem, using the lower bound derived by Tu [34], which is limited to a geometric decomposition of the domain Ω on every decomposition level. In particular:

Assumption 8 *Each subdomain Ω_i^ℓ , $\ell = 0, \dots, L-1$ and $i = 1, \dots, N^\ell$ is quadrilateral. The subdomains also form on every decomposition level ℓ a quasi-uniform coarse mesh of the domain Ω with a characteristic mesh size H^ℓ .*

Let us apply the abstract results from [21] to the particular case of the lowest order Raviart-Thomas discretization and our model problem. First, define a subspace \widetilde{W}_{HB}^ℓ of discrete harmonic balanced coarse basis functions as

$$\widetilde{W}_{HB}^\ell = \left\{ w \in \widetilde{W}_H^\ell : b(w, q_0) = 0 \quad \forall q_0 \in Q_0^\ell \right\}.$$

Let $\|w\|_{a(\Omega_i^\ell)}$ be the energy norm of a balanced function $w \in \widetilde{W}_{HB}^\ell$, $\ell = 1, \dots, L-1$, restricted to subdomain Ω_i^ℓ , $i = 1, \dots, N^\ell$, and let $\|w\|_a$ be the norm obtained by piecewise integration over each Ω_i^ℓ . To apply Lemma 2 to our model problem, we need to generalize the polylogarithmic estimate from Theorem 4 to coarse levels. To this end, let $I^{\ell+1} : \widetilde{W}_{HB}^\ell \rightarrow \widetilde{U}^{\ell+1}$ be an interpolation from the level ℓ coarse degrees of freedom (i.e., level $\ell+1$ degrees of freedom) to functions in another space $\widetilde{U}^{\ell+1}$ and assume that, for all levels $\ell = 1, \dots, L-1$, and level ℓ subdomains Ω_i^ℓ , $i = 1, \dots, N^\ell$, the interpolation satisfies for all $w \in \widetilde{W}_{HB}^\ell$ and for all $\Omega_i^{\ell+1}$ the equivalence

$$c_1^\ell \|I^{\ell+1} w\|_{a(\Omega_i^{\ell+1})}^2 \leq \|I^\ell w\|_{a(\Omega_i^{\ell+1})}^2 \leq c_2^\ell \|I^{\ell+1} w\|_{a(\Omega_i^{\ell+1})}^2, \quad (36)$$

with $c_2^\ell/c_1^\ell \leq \text{const}$ bounded independently of $H^0, \dots, H^{\ell+1}$.

Remark 5 Since $I^1 = I$, the two norms are the same on $\widetilde{W}_H^0 = \widetilde{U}^1 = U^1$.

For the three-level BDDC for saddle-point problems with the lowest-order Raviart-Thomas finite element discretization in two dimensions, the result of Tu [34, Lemma 5.5], can be written in our settings for all $w \in \widetilde{W}_{HB}^1$ and for all Ω_i^2 as

$$c_1^1 \|I^2 w\|_{a(\Omega_i^2)}^2 \leq \|w\|_{a(\Omega_i^2)}^2 \leq c_2^1 \|I^2 w\|_{a(\Omega_i^2)}^2, \quad (37)$$

where I^2 is an interpolation from the coarse degrees of freedom given by the averages over substructure faces, and $c_2^1/c_1^1 \leq \text{const}$ independently of H/h .

We note that the level 2 substructures are called subregions in [34] and that $I^1 = I$. The assumption (36), along with the equivalence of seminorms on a factor space provided an equivalence of norms [21, Lemma 22], which implies by induction the following:

Lemma 3 ([21, Lemma 23]) *For all $\ell = 0, \dots, L-1$, and $i = 1, \dots, N^\ell$,*

$$c_1^\ell \|I^{\ell+1} w\|_{a(\Omega_i^{\ell+1})}^2 \leq \|w\|_{a(\Omega_i^{\ell+1})}^2 \leq c_2^\ell \|I^{\ell+1} w\|_{a(\Omega_i^{\ell+1})}^2, \quad \forall w \in \widetilde{W}_{HB}^\ell, \forall \Omega_i^{\ell+1}, \quad (38)$$

with $c_2^\ell/c_1^\ell \leq C^\ell$, independently of $H^0, \dots, H^{\ell+1}$.

Next, using Lemma 3, we generalize the polylogarithmic estimate from Theorem 4 to coarse levels.

Lemma 4 *For all substructuring levels $\ell = 1, \dots, L-1$,*

$$\|(I - P^\ell)E^\ell w^\ell\|_a^2 \leq C^\ell \left(1 + \log \frac{H^\ell}{H^{\ell-1}}\right)^2 \|w^\ell\|_a^2, \quad \forall w^\ell \in \widetilde{W}_B^\ell. \quad (39)$$

Theorem 9 *The Multilevel BDDC for the model saddle-point problem in 2D with face coarse functions satisfies the condition number estimate*

$$\kappa \leq \prod_{\ell=1}^{L-1} C^\ell \left(1 + \log \frac{H^\ell}{H^{\ell-1}}\right)^2.$$

Proof The proof follows from Lemmas 2 and 4, with $\omega^\ell = C^\ell \left(1 + \log \frac{H^\ell}{H^{\ell-1}}\right)^2$. \square

Remark 6 For $L = 3$ we recover the estimate by Tu [34, Theorem 6.2].

Corollary 1 *In the case of uniform coarsening, i.e. with $H^\ell/H^{\ell-1} = H/h$ and the same geometry of decomposition on all levels $\ell = 1, \dots, L-1$, we get*

$$\kappa \leq C^{L-1} (1 + \log H/h)^{2(L-1)}. \quad (40)$$

7 Numerical experiments

Numerical examples are presented for a Darcy's problem on a square domain in 2D discretized by the lowest order quadrilateral Raviart-Thomas finite elements (RT0). A square domain was uniformly divided into substructures with fixed $H^\ell/H^{\ell-1}$ ratio on each level ℓ . The boundary conditions did not allow any flux across the boundary. The right-hand side was given by a unit source and sink in two distant corners of the domain, so that the compatibility condition (5) was satisfied. The method has been implemented in Matlab and for the preconditioned gradients we have used zero initial guess and stopping criterion for a relative residual tolerance of 10^{-6} . The results for different coarsening ratios $H^\ell/H^{\ell-1}$ (the relative subdomain size) and varying number of outer iterations given by the number of levels L , are reported in Table 1. For each L , there were $L-1$ outer iterations ℓ , i.e., $\ell = 1, \dots, L-1$, consisting of the three steps described in Algorithms 2 and 5. In the third step the flux correction was computed by PCG with the $(\ell+1)$ -level BDDC preconditioner.

From the results in Table 1 we can observe that with increasing number of levels, the condition number grows as predicted by Theorem 9 and in particular by formula (40). Also, it appears that for a fixed number of levels the condition number grows only mildly with increasing $H^\ell/H^{\ell-1}$ ratio.

Finally, we remark that numerical experiments with the three-level method carried out by Tu [34] indicated that the method is independent of jumps in coefficients, if they are aligned with the decomposition into subdomains on the top level. Because we feel that this prevents a practical use of the proposed method for a realistic reservoir simulations with randomly distributed coefficients with large jumps, we will address this issue in a separate study.

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Table 1 The number of (inner) PCG iterations with Multilevel BDDC preconditioner for different relative subdomain sizes $H^\ell/H^{\ell-1}$ and different number of levels (scales) L . For each level $\ell = 1, \dots, L-1$, nsub is the number of subdomains, n is the total number of degrees of freedom, n_Γ is the number of degrees of freedom on the interfaces, iter is the number of PCG iterations with the $(\ell+1)$ -level BDDC preconditioner with tol. 10^{-6} , and cond is the condition number estimate from the Lánczos sequence in conjugate gradients.

L	ℓ	nsub	n	n_Γ	iter	cond
$H_\ell/H_{\ell-1} = 3$						
2	1	9	261	36	4	1.22
3	2	9	225	36	3	1.14
	1	81	2241	432	8	2.07
4	3	9	225	36	3	1.14
	2	81	2133	432	7	1.84
	1	729	19,845	4212	11	3.48
5	4	9	225	36	3	1.14
	3	81	2133	432	7	1.83
	2	729	19,521	4212	10	3.09
	1	6561	177,633	38,880	14	5.98
$H_\ell/H_{\ell-1} = 4$						
2	1	16	800	96	6	1.94
3	2	16	736	96	5	1.73
	1	256	12,416	1920	10	3.45
4	3	16	736	96	5	1.72
	2	256	12,160	1920	9	3.11
	1	4096	197,120	32,256	14	6.62
$H_\ell/H_{\ell-1} = 6$						
2	1	36	3960	360	9	2.57
3	2	36	3816	360	9	2.30
	1	1296	140,400	15,120	13	5.60
$H_\ell/H_{\ell-1} = 8$						
2	1	64	12,416	896	10	3.00
3	2	64	12,160	896	10	2.72
	1	4096	787,456	64,512	17	7.46
$H_\ell/H_{\ell-1} = 16$						
2	1	256	197,120	7680	13	4.09
$H_\ell/H_{\ell-1} = 32$						
2	1	1024	3,147,776	63,488	15	5.25

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